An Approximation Method for Risk Aggregations and Capital Allocation Rules Based on Additive Risk Factor Models

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Abstract

This paper proposes the use of convex lower bounds as approximation to evaluate the aggregation of risks, based on additive risk factor models in the multivariate generalized Gamma distribution context. We consider two types of additive risk factor model. In Model 1, the risk factors that contribute to the aggregation are deterministic. In Model 2, we consider contingent risk factors. We work out the explicit formulae of the convex lower bounds, by which we propose an analytical approximate capital allocation rule based on the conditional tail expectation. We conduct stress tests to show that our method is robust across various dependence structures.

Key-words: risk aggregation, convex lower bound, capital allocation, approximation, generalized Gamma distribution.

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1 Introduction

Risk aggregation is a pervasive issue in finance and insurance. In the context of additive risk factor models, the quantities of interest related to aggregation are determined by individual risk factors and interdependence within the summation. In the majority of cases, the aggregation is influenced by one or more common factors—such as geography, inflation or economic environment—as well as certain idiosyncratic characteristics. Due to such multivariate complexity, the joint distribution of the summands is usually out of reach and the probability distribution of the aggregation is either too difficult to specify or cumbersome to work with. Consequently, abundant literature delves into finding accurate approximations that are practical and tractable to compute the quantities of interests.

One successful approximation is the aggregation's lower bounds in the sense of convex order. In particular, the so-called "maximal variance" convex lower bound has been shown both precise and tractable, especially for the sum of lognormal distributed random variables; see for instance, Vanduffel et al. (2008). Good performance of such a convex lower bound method is reasonable. On the one hand, a comonotonicity dependence structure is inherent to the convex lower bound, facilitating analytical computation of many risk measures, such as Value-at-Risk (VaR), Tail Value-at-Risk (TVaR) and Conditional Tail Expectation (CTE) etc. On the other hand, the "maximal variance" property ensures a global optimal (precise) approximation to the true values of the quantities of interest. Reviews on comonotonicity and convex order can be found in Dhaene et al. (2002a, 2002b) and Deelstra et al. (2011).

Thanks to this preferable tractability under lognormal distribution, the convex lower bound approximation method has been widely applied in financial valuations. Rogers and Shi (1995) and Dhaene et al. (2002b) considered its applications in derivative pricing and hedging. Dhaene et al. (2012) obtained approximate solutions in a multiperiod portfolio selection problem under the Black-Scholes type market. Dhaene et al. (2008) proposed a CTE-based capital allocation rule based on the convex lower bound. This method has also been applied in life insurance, see for instance, Denuit and Dhaene (2007). More recently, Deelstra et al. (2014) proposed the use of convex lower bounds as control variates to improve the efficiency of Monte Carlo simulation in Asian option's pricing. In contrast, this method rarely appears in actuarial models, such as the collective risk model and the individual risk model (Dickson (2005)). This is probably because deriving the analytical convex lower bounds is typically a hard job for general distributions. As an extension to the results in the literature, we address the convex lower bound approximation in a more general distribution family in this paper, namely the generalized Gamma distribution.

The generalized Gamma distribution is a probability law for non-negative random variables. It includes many well-known probability distributions that are frequently used in modelling risks as special cases such as exponential distribution, Gamma distribution, Weibull distribution and lognormal distribution etc. Following the aforementioned works, we develop the convex lower bound approximate method for the sum of generalized Gamma distributed random variables based on additive risk factors, by which the summands are dependent via common risk factors. There are two models in our framework. In the first model (Model 1), the risk factors that contributes to aggregation are deterministic. By contrast, the second model (Model 2) considers contingent risk factors, i.e., we further introduce random binary indices (Bernoulli random variables) to the system, which randomly allocates whether a risk factor contributes to aggregation. Note that these random indices may be dependent on each other in our model. Obviously, Model 2 is more general than Model 1 and could incorporate individual risk models and credit risk models. In particular, the system has additional randomness from the contingency in Model 2.

We make several contributions in this paper. First, we explicitly derive the convex lower bound for the sum of *generalized Gamma* random variables under Model 1. Our framework covers the models of Mathai and Moschopoulos (1991, 1992) and Furman (2008) as special cases. Nadarajah (2008) already commented that the probability law of the sum of independent Weibull random variables (a special case in our framework) remains unknown and is difficult to work with. We provide an approach to tackle the related problems. Particularly, we show that our convex lower bound is again generalized Gamma distributed when the individual risks follow Weibull distribution. This underpins the work of Filho and Yacoub (2006), in which the authors proposed to approximate the sum of Weibull distributed random variables by a generalized Gamma distributed random variable via the moment matching method. Moreover, we propose an alternative convex lower bound when some specific dependence structures are imposed, namely the ones appearing in Mathai and Moschopoulos (1991, 1992). This alternative bound creates additional elbow room for approaching various practical problems efficiently. Particularly, by virtue of the analytical valuation formulae, these bounds can be used together as (multi-)control variates for improving the speed of Monte Carlo simulation schemes.

Second, we derive analytical formulae of frequently-used risk measures (i.e., the Value-at-Risk and the Conditional Tail Expectation) based on the convex lower bounds in Model 1. We also work out an approximate capital allocation rule following Dhaene et al. (2008). Note that our approximate allocation rule is explicit. Hence, it does not suffer from the computational efforts and errors in simulation. We present numerical results that suggest our method provide sound approximations.

Third, we further extend our convex lower bound approximation method to Model 2. Still, we find our approach is valid in this more general context, as is the approximate capital allocation rule. As an example of our results, we present the capital allocation rules of the dependent multi-business lines for insurance companies. Moreover, we implement stress tests with respect the contingency of the random indices. We consider different levels of joint default probability to show that our approximate capital allocation rule is very robust. Therefore, compared with the discretization methods that are usually employed to tackle the aggregation of contingent random variables, our approach could be preferable due to its analyticity and robustness.

The rest of this paper is organized as follows. Section 2 sets up the models. Section 3 briefly reviews concepts of convex orders and motivates our approach. Section 4 and Section 5 consist of the main contributions of this paper. Specifically, Section 4 studies the convex lower bound approach and the capital allocation rule under Model 1 and Section 5 extends these works to Model 2. Stress-tests are conducted in Section 5 to verify the robustness of our approach. Numerical results are presented accordingly in context. Section 6 concludes the paper.

2 Model Setup

We aim to find an approximation with computational tractability to

$$S = \sum_{i=1}^{n} Z_i = \sum_{i=1}^{n} \lambda_i X_i^{\frac{1}{\nu_i}}, \tag{1}$$

where the X_i are dependent Gamma random variables with scale parameter 1 and shape parameter γ_i (notation $X_i \sim \Gamma(\gamma_i, 1)$). In fact, by (1), Z_i is the so-called generalized Gamma distribution (notation $Z_i \sim GG(\gamma_i, \lambda_i, \nu_i)$); see Stacy (1962). The generalized Gamma distribution is a key probability laws in statistics with widespread applications. Ashkar et al. (1988) remarked that its density function can assume many possible forms that are commonly encountered in hydrology. The generalized Gamma distribution also appears useful to address econometric problems caused by data skewness in health care applications; see Manning et al. (2005). Its applications in engineering and reliability are numerous as well; see Agarwal and Kalla (1996) and Sagias and Mathiopoulos (2005) amongst others. Details on the probabilistic properties of the generalized Gamma distribution are presented in the appendix We consider two types of additive risk factor model to construct the dependence for X_i in this paper.

2.1 Model 1: the aggregation of deterministic risk factors

Let $Y_j \sim \Gamma(\delta_j, 1)$, $j = 1, 2, ..., m$ be independent Gamma distributed risk factors with shape parameters $\delta_j > 0$, and $A := (a_{ij})_{n\times m}$ be a $n \times m$ matrix with binary constant $a_{ij} := 0$ or 1. We set

$$\begin{aligned}
\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}_{n\times 1} &= \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ a_{12} & \cdots & a_{2m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}_{n\times m} \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{pmatrix}_{m\times 1}, \text{ i.e.,} \qquad (2) \\
X_i &= \sum_{j=1}^{m} a_{ij} Y_j, i = 1, 2, ..., n. \qquad (3)
\end{aligned}$$

Clearly $X_i \sim \Gamma(\gamma_i, 1)$, $\gamma_i = \sum_{j=1}^{m} a_{ij}\delta_j$ thus we obtain $Z_i := \lambda_i X_i^{\frac{1}{\nu_i}} \sim GG(\gamma_i, \lambda_i, \nu_i)$ that are dependent via the common components $a_{ij}Y_j$.

Dependence (2) based on the risk factor model is not new in the literature. Mathai and Moschopoulos (1991, 1992) discussed the statistical properties of S in a special case of multivariate Gamma distribution. Namely, in our framework, they take $\nu_i \equiv 1$ and

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{n\times(n+1)}. \qquad (4)$$

In this case, Z_i are correlated by a unique common factor Y_1 whereas the other $Y_j, j = 1, 2, ..., m$ enclose idiosyncratic risks. It is believed to be suitable for modelling reliability and waiting-time problems. Alai et al. (2013) studied the survival analysis under this dependence structure. Furman and Landsman (2005) and Furman (2008) used this special model to model insurance loss and worked out the risk capital decomposition rules. More recently, Su and Furman (2016) defined a multivariate Pareto distribution for Gamma random variables constructed by (2) and presented some distributional properties. Note that in Model 1, whether a risk factor Y_j contributes to the risk X_i (thus Z_i) is deterministic, i.e., the entries of matrix A are binary constants.

2.2 Model 2: the aggregation of contingent risk factors

As a matter of fact, financial and actuarial losses are contingent in a majority of cases. This implies that there is not only a random variable to describe the severity of a loss but also a probabilistic rate of whether the loss occurs or not. For example, the "default rate" in credit risk models and the "claim rate" in individual risk models are such Bernoulli random variables. In this regard, we further randomize the dependence matrix by assigning a default rate p_{ij} to each binary, i.e., set each matrix entry as a Bernoulli random variable and redenote it $\widehat{A} := (\widehat{a}_{ij})_{n\times m}$, $\widehat{a}_{ij} \sim Ber(p_{ij})$ with

$$\Pr(\widehat{a}_{ij} = 1) = p_{ij} \text{ and } \Pr(\widehat{a}_{ij} = 0) = 1 - p_{ij}; 0 \leq p_{ij} \leq 1. \tag{5}$$

Thus, we build up the model of contingent risk factors as

$$\begin{aligned} S &= \sum_{i=1}^{n} Z_i = \sum_{i=1}^{n} \lambda_i X_i^{\frac{1}{\nu_i}}, \\ \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}_{n\times 1} &= \begin{pmatrix} \widehat{a}_{11} & \cdots & \widehat{a}_{1m} \\ \widehat{a}_{12} & \cdots & \widehat{a}_{2m} \\ \vdots & \ddots & \vdots \\ \widehat{a}_{n1} & \cdots & \widehat{a}_{nm} \end{pmatrix}_{n\times m} \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{pmatrix}_{m\times 1}, \text{ i.e.,} & (6) \\ X_i &= \sum_{j=1}^{m} \widehat{a}_{ij} Y_j, \widehat{a}_{ij} \sim Ber(p_{ij}), 0 \leq p_{ij} \leq 1, i = 1, 2, ..., n. & (7) \end{aligned}$$

Note that unlike in Model 1, $\widehat{a}_{ij}$ is not a constant but a random variable and (6) covers (2) as a special case (if $p_{ij} = 0$, then $\widehat{a}_{ij} \equiv 0$; if $p_{ij} = 1$, then $\widehat{a}_{ij} \equiv 1$).

The randomized matrix $\widehat{A}$ significantly extends our model. (6) encompasses the classic individual risks models and credit risks models, in which Y_i are individual losses and Z_i can be understood as insurance business lines or sub-credit portfolios. It is possible that some claims could trigger simultaneous losses for several insurance business lines such as a traffic accident that causes both car damage and driver injury. Likewise, in credit risk management, one default may cause loss for several financial institutions.

Furthermore, the default rates may be implicitly dependent on each other. To account for this issue, we further impose a dependence structure for $\widehat{A}_{ij}$ across different business lines. More specifically, we assume that N independent random indices $I_1, ..., I_N$ ($I_k \sim Ber(q_k)$, $0 \leq q_k \leq p_{ij}$, $k = 1, 2, ..., N$ for all i, j) are shared by the default rates and each rate also carries its own individual index U_{ij} ($U_{ij} \sim Ber(q_{ij}), 0 \leq q_{ij} \leq p_{ij}$) such that

$$\widehat{a}_{ij} = U_{ij} \prod_{k_{ij}} I_{k_{ij}}, p_{ij} = q_{ij} \prod_{k_{ij}} q_{k_{ij}} \tag{8}$$

where k_{ij} is a collection of positive integers picked from the set $\{1, 2, ..., N\}$. In other words, each $\widehat{a}_{ij}$ is impacted by a collection of common indices and its own idiosyncratic indices.

According to the setting of (8), $\widehat{a}_{ij}$ are dependent on each other. In practice, I_k can be regarded as the common indices driving the defaults of loss such as the economic environment, regional events, etc. In extreme scenarios, defaults could happen simultaneously. However, a business unit(Z_i)

could stay uninfluenced from some risk factors due to its well idiosyncrasy (U_{ij}). Note that in (8), for a specific $\widehat{a}_{ij}$, some common indices (I_k) can be absent. Moreover, for a given p_{ij}, the higher q_{ij} is, the more $\widehat{a}_{ij}$ are interconnected. In the extreme case that $q_{ij} = 1$ for all (i, j), the system is dominated by the common indices and no idiosyncratic index can help the business unit to stay unaffected by the risk factor of the losses. We present two specific examples to illustrate Model 1 and Model 2.

Example 1

$$\begin{aligned} X_1 &= Y_1 + Y_2; X_2 = Y_1 + Y_3; X_3 = Y_1 + Y_4; \\ Z_i &= \lambda_i X_i^{1/\nu_i}, i = 1, 2, 3. \\ S &= Z_1 + Z_2 + Z_3, \end{aligned}$$

Example 2

$$\begin{aligned} X_1 &= \widehat{a}_{11}Y_1 + \widehat{a}_{12}Y_2; \widehat{a}_{11} = I_1U_{11}; \widehat{a}_{12} = I_2U_{12}; \\ X_2 &= \widehat{a}_{21}Y_1 + \widehat{a}_{23}Y_3; \widehat{a}_{21} = I_1U_{21}; \widehat{a}_{23} = I_2U_{23}; \\ X_3 &= \widehat{a}_{31}Y_1 + \widehat{a}_{34}Y_4; \widehat{a}_{31} = I_1U_{31}; \widehat{a}_{34} = I_2U_{34}. \\ S &= Z_1 + Z_2 + Z_3, Z_i = \lambda_i X_i^{1/\nu_i}, i = 1, 2, 3. \end{aligned}$$

Remark 1

1. *Example 1 is in the form of (4) which sets up as Model 1 whereas Example 2 is built in line with Model 2. By contrast with Example 1, risk factors Y_j carry Bernoulli distributed random variables $\widehat{a}_{ij}$ in Example 2. More specifically, $\widehat{a}_{\cdot 1}$ are dependent via index I_1 and $\widehat{a}_{\cdot 2}$ are dependent via index I_2. Specially, when $\Pr(\widehat{a}_{ij} = 1) = 1$, Example 2 collapses to Example 1.*

2. *Note that the distribution of the aggregation S and X_i are absolutely continuous in Model 1 (hence also in Example 1) whereas they are NOT in Model 2 (nor in Example 2). In particular, $\Pr(X_1 = X_2 = ... = X_n = 0) > 0$ is possible in the context of Model 2.*

3. *The common shock Y_1 causes losses to all business units $X_i, i = 1, 2, 3$. However, some X_i can be free from the shock because of its idiosyncratic index U_{ij}. The idiosyncratic shocks Y_2, Y_3* and *Y_4 are also linked via index I_2. When S is regarded as the aggregation of multiple business lines and $\widehat{a}_{ij}Y_i$ are individual contingent losses of the corresponding business unit $Z_i(X_i)$, all business lines are tied with each other due to the common shock on Y_1. On the other hand, idiosyncratic shocks $\widehat{a}_{12}Y_2$, $\widehat{a}_{23}Y_3$, $\widehat{a}_{34}Y_4$ could add different amounts of losses to each business line. Moreover, λ_i and ν_i can be understood as the rescaling of the losses, which further extends the applicability of Model 2.*

3 Convex Lower Bounds and Allocation Rules

3.1 Conditional expectation as convex lower bounds

Using convex lower bounds to approximate a sum of random variables can be traced back to Rogers and Shi (1995), who developed this framework in the lognormal distribution context. Highly accurate convex lower bounds for lognormal distribution have been developed in Dhaene et al. (2002b) and Vanduffel et al. (2008), amongst others. We first briefly revise the convex order in this section; more details can be found in Dhaene et al. (2006).

A random variable X is said to precede another random variable Y in the convex order (notation $X \leq_{cx} Y$) if and only if the stop-loss premium of X is lower than Y and they have equal expectations[1]. Formally,

$$X \leq_{cx} Y \Leftrightarrow \begin{cases} E[X] = E[Y], \\ E\left[(X-d)^+\right] \leq E\left[(Y-d)^+\right], \qquad -\infty < d < +\infty. \end{cases}$$

where $(x-d)^+ = \max(x-d, 0)$. Note that $X \leq_{cx} Y$ if and only if $E[u(X)] \leq E[u(Y)]$ for all convex functions $u(x)$. Then, it is easy to see that $E[S|\Lambda] \leq_{cx} S$ for any random variable Λ due to the well-known Jensen's inequality.

The priori $E[S|\Lambda]$ is of great interest. First, by appropriate selection of the conditioning random variable Λ, $E[S|\Lambda]$ can be "very close" to S. Second, $E[S|\Lambda]$ could provide tractability because the inherent n-dimensionality of S reduces to one-dimension (through Λ). Hence for suitable Λ choice, it is possible that $E[S|\Lambda]$ facilitates explicit calculations of the quantities of interest while approximating S to a close extent. Moreover, $E[S|\Lambda]$ offers lower bounds for the so-called concave distortion risk measures; see Dhaene et al. (2006). In particular, we have this holds for the Conditional Tail Expectation (CTE), defined as

$$\begin{aligned} \mathrm{CTE}_p[X] \quad &: \;= E[X|X > Q_p[X]] = \frac{1}{1-p}\int_p^1 Q_q[X]dq, \quad 0 < p < 1, \\ Q_q[X] \quad &: \;= \inf\{x \in \mathbb{R} \mid \Pr(X \leq x) \geq q\}, 0 < q < 1. \end{aligned}$$

$Q_q[X]$ is the quantile of X, also known as the Value-at-risk when it is used as a risk measure. Using notation $S^l := E[S|\Lambda]$, we have $\mathrm{CTE}_p[S^l] \leq \mathrm{CTE}_p[S]$. When S^l and S are "close", so are the $\mathrm{CTE}_p[S^l]$ and $\mathrm{CTE}_p[S]$. Note that

$$S^l = \sum_{i=1}^{n} E[Z_i|\Lambda]$$

is a random variable that (only) relies on Λ. By choosing Λ such that $E[Z_i|\Lambda]$ is an increasing with respect to Λ, all $E[Z_i|\Lambda]$ are comonotonic, which facilitates the so-called comonotonic decomposition for CTE and VaR (Dhaene

[1] In the remainder of this paper, all expectations are tacitly assumed to exist.

et al. (2006)):

$$\begin{aligned} \mathrm{CTE}_p[S^l] &= \sum_{i=1}^{n} \mathrm{CTE}_p[E[Z_i|\Lambda]]; \\ Q_p[S^l] &= \sum_{i=1}^{n} Q_p[E[Z_i|\Lambda]]. \end{aligned} \tag{9}$$

It is easy to see that (9) can be computed directly via the additivity of the marginals. When $Z_i, i = 1, 2, .., n$ follow joint lognormal distribution, explicit formulae of (9) are derived in Dhaene et al. (2006) and Vanduffel et al. (2008). Their results show that (9) serves as a highly accurate approximation to its couterparts of S.

3.2 Capital allocation rule via CTE

Financial institutions need to allocate an available capital amount, K, across various constituents, e.g. business lines, types of exposure, territories or even individual products in an insurance portfolio. On the one hand, there is a need to redistribute the cost of holding capital across various constituents such that it is equitably transferred back to depositors or policyholders in the form of charges; on the other hand, capital allocation provides a useful device of assessing and comparing the performances of different lines by determining the return on allocated capital for each. More specifically, a financial institution needs to assign a proportional capital K_i to each business unit such that

$$K = \sum_{i=1}^{n} K_i$$

and K_i is a buffer against the possible loss of Z_i under certain risk measure. In the literature, the Conditional Tail Expectation (CTE) has been extensively discussed as a suitable measure of risk for setting capital requirements; see for instance Wang (2002) and Dhaene et al. (2012) amongst others. For continuous random variables, CTE is not only a coherent risk measure (i.e., it ensures the preferable pooling effect for business units) but it also takes the dependence structure among random variables into account (i.e., the business unit attributes more if its conditional expectation is larger given the total loss is large). More specifically, Note that

$$\mathrm{CTE}_p\left[S\right] = \sum_{i=1}^{n} E[Z_i|S > Q_p\left[S\right]]; \tag{10}$$

thus we have the *CTE capital allocation rule* as

$$K_i = K \frac{\mathrm{E}[Z_i|S > Q_p\left[S\right]]}{CTE_p\left[S\right]}. \tag{11}$$

The CTE capital allocation principle is consistent with the Euler allocations in Tasche (2004). In particular, when K is taken as $\mathrm{CTE}_p[S]$, K_i collapses to the "contributions to expected shortfall" in Overbeck (2000). Panjer (2002) provided a closed-form expression for this allocation rule when the risks are multivariate normally distributed, and Landsman and Valdez (2003) extended Panjer's result to the case where risks are multivariate elliptically distributed. The CTE in the proportion can be replaced by other risk measures. For instance, when the VaR is applied, we have the well-known haircut allocation principle; see also Dhaene et al. (2010). More generally, Furman and Zitikis (2008) proposed a weighted risk capital allocation framework that incorporates the proportional allocation rule (11). In a recent paper by Furman, Kuznetsov and Zitikis (2017), a condition was worked out that represents K_i in terms of the corresponding risk measure of S. However, in our framework, the explicit formula of $\mathrm{CTE}_p[S]$ is out of reach due to the difficulty in estimating the distribution of S.

4 Main Results for Model 1

This section works with Model 1. Model 1 only deals with the aggregation of deterministic risk factors; i.e., each a_{ij} is a binary constant. We derive a general convex lower bound that is valid for any given A, and an alternative convex lower bound for a specific form of A as (4). As an application, we further derive the CTE-based approximate capital allocation rules via these convex lower bounds accordingly.

4.1 A general convex lower bound

We first propose a general convex lower bound $E[S|\Lambda]$ for S with conditioning random variable Λ given as

$$\Lambda = \sum_{j=1}^{m} Y_j,$$

which is the sum of all risk factors. We label this convex lower bound as "gLB". In order to have explicit expressions of $E[S|\Lambda]$ we first work out the conditional density of $X_i|(\Lambda = z)$. This is the topic of the following lemma.

Lemma 1 *Consider an additive risk factor model as Model 1 with independent risk factors $Y_j \sim \Gamma(\delta_j, 1)$, $j = 1, 2, ..., m$. A is a matrix defined in (2) and X_i is defined as (3). Let $\Lambda = \sum_{j=1}^{m} Y_j$. We find that for $i = 1, 2, ..., n$,*

$$X_i|(\Lambda = z) \stackrel{d}{=} zB_i \qquad z > 0. \tag{12}$$

where B_i is a Beta($\beta_i, \beta-\beta_i$) distributed random variable and $\beta_i = \sum_{j=1}^{m} a_{ij}\delta_j$, $\beta = \sum_{j=1}^{m} \delta_j$.

Proof for Lemma 1 is presented in Appendix. We compute "gLB" explicitly in the following theorem.

Theorem 2 *Consider an additive risk factor model as Model 1 with independent risk factors $Y_j \sim \Gamma(\delta_j, 1), j = 1, 2, ..., m$. A is a matrix defined in (2) and X_i is defined in (3). Let $\Lambda = \sum_{j=1}^{m} Y_j$. We find that*

$$E[S|\Lambda] = \sum_{i=1}^{n} c_i.\Lambda^{\frac{1}{\nu_i}}, \tag{13}$$

with coefficients c_i given by

$$c_i = \lambda_i E[B_i^{\frac{1}{\nu_i}}] = \lambda_i \frac{\Gamma(\beta)\Gamma(\beta_i + \frac{1}{\nu_i})}{\Gamma(\beta_i)\Gamma(\beta + \frac{1}{\nu_i})}$$

and $\Lambda \sim \Gamma(\beta, 1)$, where $\beta_i = \sum_{j=1}^{m} a_{ij}\delta_j$ and $\beta = \sum_{j=1}^{m} \delta_j$.

Proof. From (12) we immediately obtain that,

$$E[\lambda_i.X_i^{\frac{1}{\nu_i}} \mid \Lambda = z] = \lambda_i E[B_i^{\frac{1}{\nu_i}}].z^{\frac{1}{\nu_i}},$$

and thus also that

$$E[S|\Lambda] = \sum_{i=1}^{n} c_i \Lambda^{\frac{1}{\nu_i}}, with\ c_i = \sum_{i=1}^{n} \lambda_i E[B_i^{\frac{1}{\nu_i}}].$$

By invoking well-known expressions for the moments of a beta distribution the result readily follows. ∎

From Theorem 2, one observes that all summands $c_i \Lambda^{\frac{1}{\nu_i}}$ $(i = 1, 2, ..., n)$ of $E[S|\Lambda]$ are increasing in Λ. In other words, $E[S|\Lambda]$ is a comonotonic sum with respect to Λ facilitating tractable calculations of many quantities of interest. In particular, the probability law of $E[S|\Lambda]$ is determined via its quantiles function,

$$Q_p[E[S|\Lambda]] = \sum_{i=1}^{n} c_i.(Q_p[\Lambda])^{\frac{1}{\nu_i}}, 0 < p < 1.$$

Note that each $z_i.Z^{\frac{1}{\nu_i}}$ is again generalized Gamma distributed (see Definition 1 in appendix). Furthermore, $E[S|\Lambda]$ is statistically the "best unbiased approximation" for S based on Λ. Hence the conditioning random variable $\Lambda = \sum_{j=1}^{m} Y_j$ contains useful information about S, which suggests that $E[S|\Lambda]$ could indeed be "close" to S. This underpins the idea of Filho and Yacoub (2006), who used a generalized Gamma distributed random variable to moment-match the distribution function of a sum of Weibull distributed random variables. Table 1 presents numerical results for the approximation of S^l to S.

4.2 Alternative convex lower bounds based on the common factor

When dependence is constructed via a unique common factor Y_1 (e.g. A is taken as (4)), which can be understood as a sole systematic risk factor, the aggregate risk S could be dominated by this systematic risk especially in extreme scenarios. This suggests that using Y_1 to capture the behavior of S; i.e., we can set $\Lambda = Y_1$[2]. We label this lower bound as "aLB".

Theorem 3 *Consider an additive risk factor model as Model 1 with independent risk factors $Y_j \sim \Gamma(\delta_j, 1), j = 1, 2, ..., m$. A is a matrix defined in (4) and X_i is defined in (3). Let $\Lambda = Y_1$. We find that*

$$E[S|\Lambda] = \sum_{i=1}^{n} \lambda_i \Lambda^{\delta_{i+1}+\frac{1}{\nu_i}} U(\delta_{i+1}, \delta_{i+1} + \frac{1}{\nu_i} + 1, \Lambda), \tag{14}$$

where U is the confluent hypergeometric function of the second kind[3].

Proof. As the common Gamma component Y_1 is independent from the other $Y_{i+1}, i = 1, 2, ..., n$, we have

$$E[X_i^{\frac{1}{\nu_i}}|Y_1 = y_1] = E[(Y_{i+1} + y_1)^{\frac{1}{\nu_i}}|Y_1 = y_1] = y_1^{\frac{1}{\nu_i}} E[(1 + Y_{i+1}/y_1)^{\frac{1}{\nu_i}}].$$

Since $Y_{i+1} \sim \Gamma(\delta_{i+1}, 1)$, it follows that

$$\begin{aligned} E[(1 + Y_{i+1}/y_1)^{\frac{1}{\nu_i}}] &= \frac{y_1^{\delta_{i+1}}}{\Gamma(\gamma_{i+1})} \int_0^\infty (1 + y)^{\frac{1}{\nu_i}} y^{\delta_{i+1}-1} e^{-y_1 y} dy \\ &= y_1^{\delta_{i+1}} U(\delta_{i+1}, \delta_{i+1} + \frac{1}{\nu_i} + 1, y_1) \end{aligned}$$

This implies,

$$E[S|\Lambda] = \sum_{i=1}^{n} \lambda_i E[X_i^{\frac{1}{\nu_i}}|Y_1] = \sum_{i=1}^{n} \lambda_i (Y_1)^{\delta_{i+1}+\frac{1}{\nu_i}} U(\delta_{i+1}, \delta_{i+1} + \frac{1}{\nu_i} + 1, Y_1),$$

where U is the confluent hypergeometric function of the second kind. ■

We observe that $E[S|Y_1]$ is also increasing in Y_1 hence $E[S|\Lambda]$ in (14) is a comonotonic sum with respect to Y_1. Then, the distribution of aLB is also directly determined by its quantile function.

$$Q_p[E[S|Y_1]] = \sum_{i=1}^{n} \lambda_i Q_p[Y_1]^{\delta_{i+1}+\frac{1}{\nu_i}} U(\delta_{i+1}, \delta_{i+1} + \frac{1}{\nu_i} + 1, Q_p[Y_1]), 0 < p < 1.$$

[2] There can be several common Gamma risk factor. However, according to the summation property (see appendix), we consider them together as one Gamma distributed risk factor.

[3] We refer to Abramowitz and Stegun (1965) for the details of the confluent hypergeometric function.

Methods	$p = 0.05$	$p = 0.25$	$p = 0.75$	$p = 0.95$	$p = 0.99$	$p = 0.995$
MC	0.823138	1.273776	1.959038	2.440721	2.755266	2.861321
s.e.	0.00089	0.00069	0.00070	0.00098	0.00163	0.00199
gLB	0.856702	1.302239	1.939499	2.375826	2.666834	2.770184
devi.	4.078%	2.235%	0.997%	2.659%	3.210%	3.185%
aLB	0.852214	1.269346	1.952922	2.437339	2.761073	2.875895
devi.	3.532%	0.348%	0.312%	0.139%	0.211%	0.509%

Table 1: Approximations for the Quantiles of $S = \sum_{i=1}^{3} Z_i$. Z_i are dependent GG distributed random variables with $Z_i = \lambda_i X_i^{\frac{1}{\nu_i}}$, $X_i = Y_1 + Y_{i+1}$, $Y_1 \sim \Gamma(0.9, 1)$, $Y_{i+1} \sim \Gamma(0.1, 1)$, $i = 1, 2, 3$. $\nu_1 = 3$, $\nu_2 = 3.5$, $\nu_3 = 4$. $\lambda_1 = 0.5$, $\lambda_2 = 0.6$, $\lambda_3 = 0.7$. Standard error (s.e.) of the Monte Carlo simulation (MC) and the deviations (devi) from the MC results are reported accordingly. MC simulation size is 1e6, devi are presented in the percentage of the MC results.

A comparison of the approximations of aLB and gLB are presented in Table 1. We can see that both of gLB and aLB offer sound approximations with aLB seeming to outperform gLB at the upper tails where the systemic risk factor matters. However, gLB is applicable for any dependence matrix A whereas aLB requires the special form as (4).

Remark 4 *In addition to the convex lower bound approximate method, there are many other approaches for finding accurate approximations of dependent random variables aggregation such as moment matching, asymptotic methods etc. Some methods could outperform ours regarding the accuracy of approximation. For example, in a recent work of Furman, Hackmann and Kuznetsov (2017), the authors utilized the so-called generalized gamma convolution to work out an approximation of the sum of independent lognormal distributed random variables. They showed that the algorithm converges to the true sum distribution, for which their method could reasonably outperform ours (their method also fits our Model 1 when A is an identity matrix). However, our methods are applicable for additive risk factors which covers the independent case. In particular, the more positively dependent the summands are, the better our approximation performs (because it relies on the comonotonic dependence of the convex lower bound). Moreover, our method provides an easy computation to the estimation of CTE. Lastly, as we are going to show in the next section, our method is still applicable under contingent risk factors (Model 2), which significantly perplexes the distribution of aggregation.*

4.3 Approximate CTE capital allocation rule for Model 1

Based on the discussions in Section 3.2, we present the analytical approximate CTE capital allocation rules. Note that we only need to compute

$\text{CTE}_p[E[Z_i|\Lambda]]$ for all i, then $\text{CTE}_p[S^l]$ and the proportions allocation follows immediately.

Theorem 5 *Using the notations and assumptions introduced in Theorem 2, we have that for* $0 < p < 1$ *the approximation* $CTE_p[E[Z_i|\Lambda]]$ *for* $E[Z_i|S > Q_p[S]]$ *is given as,*

$$CTE_p[E[Z_i|\Lambda]] = \frac{\lambda_i \Gamma(\beta_i + 1/\nu_i)}{(1-p)\Gamma(\beta_i)} \overline{F}_i(Q_p[\Lambda]) \tag{15}$$

where $\overline{F}_i$ *is the survival function for a Gamma*$(\frac{1}{\nu_i} + \beta, 1)$ *distributed random variable.*

Proof. According to Theorem 2, we have,

$$\begin{aligned}
\text{CTE}_p[E[Z_i|\Lambda]] &= CTE_p[E[\lambda_i X_i^{\frac{1}{\nu_i}}|\Lambda]] = z_i E[\Lambda^{\frac{1}{\nu_i}}|\Lambda > Q_p[\Lambda]] \\
&= \frac{z_i}{1-p} \int_{Q_p[\Lambda]}^{+\infty} \frac{z^{\frac{1}{\nu_i}+\beta-1} e^{-z}}{\Gamma(\beta)} dz \\
&= \frac{z_i \Gamma(\beta + 1/\nu_i)}{(1-p)\Gamma(\beta)} \overline{F}_i(Q_p[\Lambda]) \\
&= \frac{\lambda_i \Gamma(\beta_i + 1/\nu_i)}{(1-p)\Gamma(\beta_i)} \overline{F}_i(Q_p[\Lambda])
\end{aligned}$$

where the z_i and Λ are as in proposition 1 and where $\overline{F}_i$ is the survival function for a Gamma$(\frac{1}{\nu_i} + \beta, 1)$ distributed random variable. ■

Table 2 provides numerical results on the approximate CTE capital allocation rule, where total capital amount K is 100 for the simplicity of proportional computations. Results suggest that our methods provide accurate approximations. Note that the approximate capital allocation rule is analytical, i.e., it does not suffer from the errors and efforts of simulations.

5 Main Results for Model 2

5.1 Convex lower bounds based on conditional expectation

This section works with Model 2. In Model 2, the aggregation involves not only risk factors (Y_j) but also random indices (I_k). The aLB approach is not applicable because common risk factors are unknown due to the contingency. By contrast, the gLB approach is still valid as it is based on all risk factors. For a given realization of $\widehat{A}$, the gLB approach is always feasible and thus the randomness from the contingent indices is simply transferred into the parametrization of gLB. Consequently, we again have a convex lower bound for the aggregation S under Model 2, labelled as "CgLB".

Method	p=0.95			p=0.99			p=0.995		
	Z_1	Z_2	Z_3	Z_1	Z_2	Z_3	Z_1	Z_2	Z_3
MC	29.62	33.33	37.05	30.10	33.32	36.58	30.28	33.31	36.41
s.e.	0.0135	0.0130	0.0126	0.0236	0.0227	0.0213	0.0297	0.0282	0.0273
CTE-gLB	29.65	33.33	37.02	30.13	33.31	36.56	30.29	33.31	36.40

Table 2: Approximate CTE capital allocation rule (CTE-gLB) for $S = \sum_{i=1}^{3} Z_i$. Z_i are dependent GG distributed random variables with $Z_i = \lambda_i X_i^{\frac{1}{\nu_i}}$, $X_i = Y_1 + Y_{i+1}$, $Y_1 \sim \Gamma(0.9, 1)$, $Y_{i+1} \sim \Gamma(0.1, 1)$, $i = 1, 2, 3$. $\nu_1 = 3$, $\nu_2 = 3.5$, $\nu_3 = 4$, $\lambda_1 = 0.5$, $\lambda_2 = 0.6$, $\lambda_3 = 0.7$. The total available capital amount K is 100. Standard error (s.e.) of the Monte Carlo simulation (MC) is reported below MC results. MC simulation size is 1e6.

Theorem 6 *Consider a contingent additive risk factor model as Model 2 with independent risk factors $Y_j \sim \Gamma(\delta_j, 1), j = 1, 2, ..., m$. Let $\widehat{A}$ be a random matrix defined in (6) with contingency $\widehat{a}_{ij} \sim Ber(p_{ij}), 0 \le p_{ij} \le 1$, which is constructed by indices $I_k \sim Ber(q_k), 0 \le q_k \le p_{ij}, k = 1, 2, ..., N.$and $U_{ij} \sim Ber(q_{ij}), 0 \le q_{ij} \le p_{ij}$ via (8). X_i is defined in (7) and $\Lambda = \sum_{j=1}^{m} Y_j$. We find that*

$$S^l_{\widehat{A}} = \sum_{i=1}^{n} C_i.\Lambda^{\frac{1}{\nu_i}}, \tag{16}$$

with coefficients C_i given by

$$C_i = \lambda_i E[B_i^{\frac{1}{\nu_i}} | \widehat{A}] = \lambda_i \frac{\Gamma(\beta)\Gamma(\eta_i + \frac{1}{\nu_i})}{\Gamma(\eta_i)\Gamma(\beta + \frac{1}{\nu_i})},$$

where $\eta_i = \sum_{j=1}^{m} \widehat{a}_{ij}\delta_j$ and $\beta = \sum_{j=1}^{m} \delta_j$. Note that both η_i and C_i are random variables and $\Lambda \sim \Gamma(\beta, 1)$.

Proof. Given any realization $\widehat{a}_{ij} = a_{ij}$, a_{ij} is either 0 or 1, from Theorem 2, it always holds that

$$E_{\widehat{A}_{ij}=a_{ij}}[S|\Lambda] = \lambda_i \frac{\Gamma(\sum_{j=1}^{m} \delta_j)\Gamma(\sum_{j=1}^{m} a_{ij}\delta_j + \frac{1}{\nu_i})}{\Gamma(\sum_{j=1}^{m} a_{ij}\delta_j)\Gamma(\sum_{j=1}^{m} \delta_j + \frac{1}{\nu_i})} \Lambda^{\frac{1}{\nu_i}}.$$

Hence, we have

$$S^l_{\widehat{A}} = \sum_{i=1}^{n} C_i.\Lambda^{\frac{1}{\nu_i}} = \lambda_i \frac{\Gamma(\beta)\Gamma(\eta_i + \frac{1}{\nu_i})}{\Gamma(\eta_i)\Gamma(\beta + \frac{1}{\nu_i})} \Lambda^{\frac{1}{\nu_i}},$$

where $\eta_i = \sum_{j=1}^{m} \widehat{a}_{ij}\delta_j$ and $\beta = \sum_{j=1}^{m} \delta_j$ due to the independence between the $\widehat{A}$ and Λ. ■

By conditioning, it is easy to see that $S^l_{\widehat{A}}$ is a lower bound for S with respect to convex order. Note that both η_i and C_i are random variables and their probability laws rely on $\widehat{A}$. This can be determined via the conditional independence between the $\widehat{A}$ and Λ. In particular, if δ_j are identical for all Y_j, then η_i follows the so-called Poisson binomial distribution and if further all p_{ij} are identical, η_i reduces to the binomial distribution.

Theorem 6 provides a comprehensive insight in Model 2. The complicated randomness inherent in the model comes from two parts, i.e., the risk factors (Y_i) and the random indices (I_k and U_{ij}). By conditioning on Λ, "CgLB" takes both parts into account separately. On the one hand, the risk factors are "projected" onto Λ via the conditional expectation, measuring the impact of the dependence among risk factors via the corresponding coefficients C_i. On the other hand, the random indices are transferred in the parametrization of C_i, accounted for independently from Λ. Consequently, "CgLB" maintains the facility and tractability of "gLB" and further allows extensions for contingent risk factor models.

In contrast to gLB and aLB, the probability law of CgLB cannot be directly determined by Λ. This is due to the fact that C_i is now a random variable, which further twists the distribution of $S^l_{\widehat{A}}$ in addition to Λ. Fortunately, by virtue of conditioning and comonotonicity, the distribution of $S^l_{\widehat{A}}$ is still reachable analytically.

$$\begin{aligned}
\Pr(S^l_{\widehat{A}} &\leq t) = \sum_{a_{ij}\in\Omega_{\widehat{A}}} \Pr(\widehat{A}_{ij} = a_{ij}) \Pr(g_{\widehat{A}}(\Lambda) \leq t), \text{ where} \\
g_{\widehat{A}}(\Lambda) &= \sum_{i=1}^{n} \lambda_i \frac{\Gamma(\beta)\Gamma(\beta_i + \frac{1}{\nu_i})}{\Gamma(\beta_i)\Gamma(\beta + \frac{1}{\nu_i})} \Lambda^{\frac{1}{\nu_i}}, \beta_i = \sum_{j=1}^{m} a_{ij}\delta_j, \beta = \sum_{j=1}^{m} \delta_j \qquad (17) \\
& \text{and } \Omega_{\widehat{A}} \text{ is the set of all realizations of } \widehat{A}.
\end{aligned}$$

Note that (17)[4] is non-decreasing and continuous w.r.t. Λ, it is easy to evaluate $\Pr(g_{\widehat{A}}(\Lambda) \leq t) = u$ by solving equations $g_{\widehat{A}}(Q_u[\Lambda]) = t$ with $0 \leq u \leq 1$ thus the quantiles of $S^l_{\widehat{A}}$ can be determined accordingly. Clearly, (16) is in a form of compound distribution, which agrees with the classical discretization method that is frequently used in actuarial models. However, discretization requires heavy computational efforts. By contrast, our method provides explicit formulae with straightforward and simple computations. Table 3 presents numerical results on the sound approximation of $S^l_{\widehat{A}}$ to S.

[4] In (17) and henceforth, for notation simplicity, we use the shorthand $\widehat{A} = a_{ij}, 0 < i < n, 0 < j < m$ to denote a realization of the random matrix $\widehat{A}$, i.e., $\widehat{a}_{11} = a_{11}, ..., \widehat{a}_{nm} = a_{nm}$, where a_{ij} are binary constants. Specifically, $\Pr(\widehat{A} = a_{ij})$ is for $\Pr(\widehat{a}_{11} = a_{11}, ..., \widehat{a}_{nm} = a_{nm})$.

Methods	$p=0.8$	$p=0.85$	$p=0.9$	$p=0.95$	$p=0.99$	$p=0.995$
MC	0.000	0.160157	0.425613	0.623892	0.956977	1.19173
s.e.	0.000	0.000973	0.001213	0.002319	0.002692	0.004186
CgLB	0.000	0.151686	0.423477	0.610018	0.932232	1.162950
devi.	0.00%	5.29%	0.50%	2.22%	2.59%	2.41%

Table 3: Approximations for the Quantiles of $S = \sum_{i=1}^{3} Z_i$ in Example 1. $Y_1 \sim \Gamma(0.9, 1)$, $Y_{i+1} \sim \Gamma(0.1, 1)$, $I_1 \sim Ber(0.5)$, $I_2 \sim Ber(0.5)$, $U_{i1} \sim Ber(0.1)$, $U_{i,i+1} \sim Ber(0.02)$, $i = 1, 2, 3$. $\nu_1 = 3$, $\nu_2 = 3.5$, $\nu_3 = 4$. $\lambda_1 = 0.5$, $\lambda_2 = 0.6$, $\lambda_3 = 0.7$. Standard error (s.e.) of the Monte Carlo simulation (MC) and the deviations (devi) from the MC results are reported accordingly. MC simulation size is 1e6, devi are presented in the percentage of the MC results.

5.2 CTE allocation rule for Model 2

In line with Theorem 5, we also develop the approximate CTE capital allocation rule for the contingent dependent losses. In Model 2, we need to adapt the formula to account for the randomness of contingency. Again, we only need to work out the approximation formula for $E[Z_i|S > Q_p[S]]$. By replacing S with $S^l_{\widehat{A}}$ in the conditioning, whose quantile at p is explicitly attainable, we have

$$
\begin{aligned}
E[Z_i|S &> Q_p[S]] \approx E[Z_i \mid S^l_{\widehat{A}} > Q_p\left[S^l_{\widehat{A}}\right]] = \frac{E[Z_i I_{S^l_{\widehat{A}} > Q_p\left[S^l_{\widehat{A}}\right]}]}{1-p} \\
&= \frac{1}{1-p} \sum_{a_{ij} \in \Omega_{\widehat{A}}} \Pr(\widehat{A}_{ij} = a_{ij}) E[Z_i I_{g_{\widehat{A}}(\Lambda) > Q_p\left[S^l_{\widehat{A}}\right]} | \widehat{A}_{ij} = a_{ij}].
\end{aligned}
$$

Note that $g_{\widehat{A}}(\cdot)$ is non-decreasing at Λ; hence given realization $\widehat{A}_{ij} = a_{ij}$, $S^l_{\widehat{A}} > Q_p\left[S^l_{\widehat{A}}\right]$ collapses to $\Lambda > Q_u[\Lambda]$ where u is a percentage that relies on the realization.

$$
\begin{aligned}
E[Z_i \quad | \quad S^l_{\widehat{A}} > Q_p\left[S^l_{\widehat{A}}\right]] &\approx \frac{1}{1-p} \sum_{a_{ij} \in \Omega_{\widehat{A}}} \Pr(\widehat{A}_{ij} = a_{ij}) E[E[Z_i|\Lambda] I_{\Lambda > Q_{u(a_{ij})}[\Lambda]} | \widehat{A}_{ij} = a_{ij}] \\
&= \frac{1}{1-p} \sum_{a_{ij} \in \Omega_{\widehat{A}}} \Pr(\widehat{A}_{ij} = a_{ij}) C_i(\widehat{A}_{ij} = a_{ij}) E[\Lambda^{\frac{1}{\nu_i}} I_{\Lambda > Q_u[\Lambda]} | \widehat{A}_{ij} = a_{ij}] \\
&= \sum_{a_{ij} \in \Omega_{\widehat{A}}} \Pr(\widehat{A}_{ij} = a_{ij}) \frac{\lambda_i \Gamma(\eta_i + \frac{1}{\nu_i})}{\Gamma(\eta_i)(1-p)} \overline{F}_i(Q_u[\Lambda])
\end{aligned}
$$

where $\eta_i = \sum_{j=1}^{m} \widehat{A}_{ij} \delta_j$ and $\overline{F}_i$ is the survival function for Gamma$(\beta + \frac{1}{\nu_i}, 1)$. Formally, we have the following Theorem.

Theorem 7 *Using the notations and assumptions introduced in Theorem 6, we have that for* $0 < p < 1$ *the approximation* $CTE_p[Z_i | S^l_{\widehat{A}} > Q_p[S^l_{\widehat{A}}]]$ *for* $E[Z_i | S > Q_p[S]]$ *is given as,*

$$CTE_p[Z_i | S^l_{\widehat{A}} > Q_p[S^l_{\widehat{A}}]] = \sum_{a_{ij} \in \Omega_{\widehat{A}}} \Pr(\widehat{A}_{ij} = a_{ij}) \frac{\lambda_i \Gamma(\eta_i + \frac{1}{\nu_i})}{\Gamma(\eta_i)(1-p)} \overline{F}_i(Q_u[\Lambda]) \quad (18)$$

where $\overline{F}_i$ *is the survival function for a Gamma*$(\frac{1}{\nu_i} + \beta, 1)$ *distributed random variable.*

In Theorem 7, the approximate CTE allocation rule (11) is already at hand. Thus we extend the approximate CTE allocation rule to the contingent risk factor models (Model 2). Note that due to the contingency, Model 2 is more difficult than Model 1. For instance, it requires more efforts to simulate Model 2 because of the additional randomness. However, our explicit approaches do not suffer from heavy computational efforts and numerical errors. We summarize numerical results on the approximate CTE allocation rule in Table 4, Table 5 and Table 6.

5.3 Stress tests for the robustness

In Model 2, the risk factors are contingent and dependent via (8). The joint default probability (the probability that all $\widehat{a}_{ij}$ are equal to 1) also varies for different sets of random indices. Thus, we further implement stress tests regarding our approximate CTE allocation rule. By doing so, we show that our approximate CTE allocation rule is not only close to the results of simulations but also very *robust* across different levels of the joint default probability. From Table 4 to Table 6, the joint default probability of the $\widehat{a}_{ij}$ arises and the approximate CTE allocation rule maintains satisfactory performances. Therefore, we can see that our method is indeed very robust.

From the technical point of view, the approximate CTE allocation rule successfully captures the dependence among the contingent risk factors while fully inheriting how the systemic risk affects the individual business units. More specifically, the convex lower bound $S^l_{\widehat{A}}$ is constructed by the conditional expectation of $\Lambda = \sum_{j=1}^m Y_j$, which contains all risk factors that describe the severity of losses whilst the contingencies of the indices are transferred into the (random) coefficients in (16). Thus, all elements that determine the allocation proportions are accounted for by $S^l_{\widehat{A}}$. Consequently, the approximate proportional allocations stay close to the simulation results despite of the level of the joint default rate.

Moreover, our method suggests a decomposition of the aggregate risk using CTE. Thanks to the robustness of the approximate allocation rule, we can identify which business unit (Z_i) contributes major risk to the aggregation. For instance, we can observe from Tables 4 to 6 that the second

Method	$p = 0.99$			$p = 0.995$		
	Z_1	Z_2	Z_3	Z_1	Z_2	Z_3
MC	24.38	31.07	44.55	24.45	32.58	42.97
s.e.	0.0224	0.0260	0.0302	0.0298	0.0322	0.0337
CTE-CgLB	24.46	31.56	43.98	24.27	32.84	42.89

Table 4: Approximate CTE capital allocation rule (CTE-CgLB) for $S = \sum_{i=1}^{3} Z_i$ in Example 2. $Y_1 \sim \Gamma(0.9, 1)$, $Y_{i+1} \sim \Gamma(0.1, 1)$, $i = 1, 2, 3$. $\nu_1 = 3$, $\nu_2 = 3.5$, $\nu_3 = 4$. $\lambda_1 = 0.5$, $\lambda_2 = 0.6$, $\lambda_3 = 0.7$. The joint default probability for common risk factor Y_1 is 5e-04. The joint default probability for idiosyncratic risk factor Y_{i+1}, $i = 1, 2, 3$ is 4e-06. The total available capital amount K is 100. Standard error (s.e.) of the Monte Carlo simulation (MC) are reported accordingly. MC simulation size is 1e6.

Method	$p = 0.99$			$p = 0.995$		
	Z_1	Z_2	Z_3	Z_1	Z_2	Z_3
MC	26.30	33.26	40.44	27.27	33.43	39.30
s.e.	0.0126	0.0125	0.0116	0.0129	0.0119	0.0113
CTE-CgLB	25.96	33.45	40.59	27.4	33.5	39.1

Table 5: Approximate CTE capital allocation rule (CTE-CgLB) for $S = \sum_{i=1}^{3} Z_i$ in Example 2. $Y_1 \sim \Gamma(0.9, 1)$, $Y_{i+1} \sim \Gamma(0.1, 1)$, $i = 1, 2, 3$. $\nu_1 = 3$, $\nu_2 = 3.5$, $\nu_3 = 4$. $\lambda_1 = 0.5$, $\lambda_2 = 0.6$, $\lambda_3 = 0.7$. The joint default probability for common risk factor Y_1 is 0.0125. The joint default probability for idiosyncratic risk factor Y_{i+1}, $i = 1, 2, 3$ is 1e-04. The total available capital amount is $K = 100$. Standard error (s.e.) of the Monte Carlo simulation (MC) are reported accordingly. MC simulation size is 1e6.

business unit Z_2 has stable contribution to total loss, with respect to different joint default levels. Additionally, we can observe how the risk transfers from one unit to another when the joint default probability varies. Note that our approximate allocation rule is explicit and free of numerical errors, which indicates reliability and limited numerical errors in conducting such sensitive analysis. Particularly, our method is helpful to compare the sensitivity of different models. Intuitively speaking, our approximate allocation rule could provide insight for how systemic risk interacts with idiosyncratic risk factors and contingencies.

6 Conclusions and Further Discussion

In this paper, we developed the convex lower bound approximation method for risk aggregations in the context of generalized Gamma distribution. Such

Method	$p = 0.99$			$p = 0.995$		
	Z_1	Z_2	Z_3	Z_1	Z_2	Z_3
MC	28.96	33.33	37.71	29.21	33.33	37.46
s.e.	0.0768	0.0213	0.078	0.0892	0.0157	0.0879
CTE-CgLB	28.78	33.33	37.89	29.09	33.33	37.58

Table 6: Approximate CTE capital allocation rule (CTE-CgLB) for $S = \sum_{i=1}^{3} Z_i$ in Example 2. $Y_1 \sim \Gamma(0.9, 1)$, $Y_{i+1} \sim \Gamma(0.1, 1)$, $i = 1, 2, 3$. $\nu_1 = 3$, $\nu_2 = 3.5$, $\nu_3 = 4$. $\lambda_1 = 0.5$, $\lambda_2 = 0.6$, $\lambda_3 = 0.7$. The joint default probability for common risk factor Y_1 is 0.05. The joint default probability for idiosyncratic risk factor $Y_{i+1}, i = 1, 2, 3$ is 0.01. The total available capital amount is $K = 100$. Standard error (s.e.) of the Monte Carlo simulation (MC) are reported accordingly. MC simulation size is 1e6.

method is preferable due to its tractability and analytical results, which facilitates straightforward computations of quantities and analyses of interest. We provided explicit solutions under the additive risk factor models based on this method. In particular, we worked out an approximate CTE-based capital allocation rule. As another distinguished contribution, we further extended our method to a model with contingent risk factors. By doing so, we significantly enhanced the applicability of our method. We showed that the approximation convex lower bound method is still valid despite of the additional randomness and complexity. Moreover, we observed that the approximate CTE capital allocation rule is very robust with respect with respect to various levels of the joint default levels.

The robustness of our method is indeed a very interesting result. On the one hand, it implies that this approximation method could provide useful insights about the model, such as dependence, systemic risks and joint default rates etc., which are crucial in determining risk aggregations. In fact, it is natural to consider the convex lower bound of risk aggregation as a counterpart for relevant problems because of the statistical mechanism (conditional expectation), especially when it is much more friendly to work with. On the other hand, the robustness of the approximate CTE capital allocation suggests the potentials of this method in related analyses (e.g. sensitivity analysis). Because there are neither heavy computational efforts nor numerical error, the results based on the approximation method are more reliable and less costly. Our method is also applicable to other models that have been frequently-used in practice to study relevant problems and analyses. For instance, the KMV model and the CreditRisk+ both share the same spirits of our Model 2. This shall be the next research topic in our further studies.

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Appendix

The generalized Gamma distribution A positive random variable X is said to follow Gamma distribution with scale parameter θ and shape parameter γ, denoted as $X \sim \Gamma(\gamma, \theta)$, if its density function writes as

$$f(x) = \frac{x^{\gamma-1}}{\theta^{\gamma}\Gamma(\gamma)} \exp(-\frac{x}{\theta}),\ x > 0,$$

where $\Gamma(\gamma)$ is the Gamma function (Abramowitz and Stegun (1965)). We provide two well-known properties of the Gamma distribution; for more details we refer to Johnson et al. (1994).

- Summation: Let $X_1 \sim \Gamma(\gamma_1, \theta)$ and $X_2 \sim \Gamma(\gamma_2, \theta)$ be independent. Then $X_1 + X_2 \sim \Gamma(\gamma_1 + \gamma_2, \theta)$.
- Scaling: Let $X \sim \Gamma(\gamma, \theta)$ and $a > 0$. Then $aX \sim \Gamma(\gamma, a\theta)$.

Johnson and Kotz (1972) show that the generalized Gamma distribution can be derived by a power transform of Gamma random variable. Indeed, let $X \sim \Gamma(\gamma, 1)$, and $Z = \lambda X^{\frac{1}{\nu}}$, then the distribution of Z follows the generalized Gamma distribution.

Definition 1 *A positive random variable Z is said to have a generalized Gamma distribution, if its density function is given as*

$$f(z) = \frac{\nu z^{\gamma\nu-1}}{\lambda^{\nu\gamma}\Gamma(\gamma)} \exp\left(-\left(\frac{z}{\lambda}\right)^{\nu}\right),\ z > 0, \tag{19}$$

with parameters $\gamma > 0$, $\lambda > 0$, $\nu > 0$.

We now list some properties of generalized Gamma distributions; see also Stacy (1962) and Johnson et al. (1994) for more details.

1. The cumulative distribution function of Z is given as,

   $$F_Z(z; \gamma, \lambda, \nu) = LG_{\left(\frac{z}{\lambda}\right)^{\nu}}(\gamma)/\Gamma(\gamma),\ z > 0.$$

   where LG is the lower incomplete gamma function, i.e. $LG_z(\gamma) = \int_0^z u^{\gamma-1}e^{-u}du$

2. The moment generating function of Z is given as,

$$M(t) = \sum_{n=0}^{\infty} \frac{(t\lambda)^n \Gamma(\gamma + \frac{n}{\nu})}{n!\Gamma(\gamma)}.$$

3. The n-th moment of Z is given as,

$$E[Z^n] = \lambda^n \frac{\Gamma(\gamma + \frac{n}{\nu})}{\Gamma(\gamma)}.$$

The proof of Lemma 1 **Proof.** Consider any $i \in \{1, 2, ..., n\}$. Then we find that

$$\begin{aligned} f_{X_i|\Lambda=z}(x) &= \frac{f_{X_i,\Lambda}(x,z)}{f_\Lambda(z)} \\ &= \frac{f_{\sum_{j=1}^m a_{ij}Y_j}(x) \cdot f_{\sum_{j=1}^m (1-a_{ij})Y_j}(z-x)}{f_\Lambda(z)}. \end{aligned}$$

Invoking the summation property and pdf of Gamma distribution,

$$\begin{aligned} f_{X_i|\Lambda=z}(x) &= \frac{\Gamma(\beta)}{\Gamma(\delta_i)\Gamma(\beta-\beta_i)} \frac{x^{\beta_i-1}(z-x)^{\beta-\beta_i-1}}{z^{\beta-1}} \\ &= \frac{1}{z} \frac{\Gamma(\beta)}{\Gamma(\beta_i)\Gamma(\beta-\beta_i)} \left(\frac{x}{z}\right)^{\beta_i-1} (1-\frac{x}{z})^{\beta-\beta_i-1}. \end{aligned}$$

But this exactly means that

$$X_i|(\Lambda = z) \stackrel{d}{=} zB_i,$$

where B_i is a Beta(β_i, $\beta - \beta_i$) distributed random variable. ■

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